

# CHARACTERIZATION OF UNITARY PROCESSES WITH INDEPENDENT INCREMENTS

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**ABSTRACT.** In this paper, we study unitary Gaussian processes with independent increments with which the unitary equivalence to a Hudson-Parthasarathy evolution systems is proved. This gives a generalization of results in [16] and [17] in the absence of the stationarity condition.

*Dedicated to Robin L. Hudson on his 70th birthday*

## 1. Introduction

In the framework of the theory of quantum stochastic calculus developed by the work of Hudson and Parthasarathy [9], HP- quantum stochastic differential equations (qsde)

$$(1.1) \quad dV_t = \sum_{\mu, \nu \geq 0} V_t L_\nu^\mu(t) \Lambda_\mu^\nu(dt), \quad V_0 = 1_{\mathbf{h} \otimes \Gamma},$$

(where the coefficients  $L_\nu^\mu(t) : \mu, \nu \geq 0$  are operators in the initial Hilbert space  $\mathbf{h}$  for almost every  $t \geq 0$  and  $\Lambda_\mu^\nu$  are fundamental processes in the symmetric Fock space  $\Gamma = \Gamma_{sym}(L^2(\mathbb{R}_+, \mathbf{k}))$  with respect to a fixed orthonormal basis (in short ‘ONB’)  $\{E_j : j \geq 1\}$  of the noise Hilbert space  $\mathbf{k}$ ) have been formulated. The conditions for existence and uniqueness of a solution  $\{V_t\}$  were studied by Hudson and Parthasarathy and many other authors. In particular when the coefficient operators  $\{L_\nu^\mu(t) : t \geq 0, \mu, \nu \geq 0\}$  are locally essentially bounded in  $\mathcal{B}(\mathbf{h})$  satisfying unitarity conditions it is observed that the solution  $\{V_t : t \geq 0\}$  is a unitary process.

In particular, in the absence of the conservation martingale, the equation take the form  $dV_t = \sum_j \{V_t L_j(t) a^\dagger(dt) - V_t L_j^*(t) a(dt)\} + V_t G(t) dt$  with the unitary condition  $\sum_j L_j^*(t) L_j(t) + 2\text{Re } G(t) = 0$  for almost every  $t \geq 0$  (Ref.[6, 9]).

In a series of earlier work (Ref.[16, 17]) it has been shown that unitary evolution on  $\mathbf{h} \otimes \mathcal{H}$  with stationary, independent increments and a Gaussian condition (where  $\mathbf{h}$  and  $\mathcal{H}$  are separable Hilbert spaces) with bounded or possibly unbounded generator ( in the

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second case, one needs some further conditions ) are unitarily isomorphic to solution of qsde of the type (1.1) with time independent coefficients.

In this article we are interested in the characterization of unitary evolutions with only independent increments on  $\mathbf{h} \otimes \mathcal{H}$  and with the assumption that the expectation evolution relative to a distinguished vector in  $\mathcal{H}$  is Lifshitz in the time variable.

The article is organized as follows: Section 2 is meant for recalling some preliminary ideas and fixing some notations on linear operators on Hilbert spaces and Section 3 collects some results associated with Hilbert space and properties of evolutions. The main results of section 3 are proved in the Appendix. Section 3 also contain the description of the unitary processes with independent increments and the assumptions on them. Section 4 is dedicated to the construction of a Hilbert space, called the noise space and operator coefficients associated with them. The HP evolution system and its minimality are discussed in Section 5 and consequently the unitary equivalence of the solution with the unitary process is proven.

## 2. Notation and Preliminaries

We assume that all Hilbert spaces in this article are complex separable with inner products which are anti-linear in the first variable. For each Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$  we denote the Banach spaces of all bounded linear operators from  $\mathcal{H}$  to  $\mathcal{K}$  and all trace class operators on  $\mathcal{H}$  by  $\mathcal{B}(\mathcal{H}, \mathcal{K})$  and  $\mathcal{B}_1(\mathcal{H})$ , respectively, and the trace on  $\mathcal{B}_1(\mathcal{H})$  by  $\text{Tr}(\cdot)$ . We note that for each  $\xi \in \mathcal{H} \otimes \mathcal{K}$  and  $h \in \mathcal{H}$ , there exists a unique vector  $\langle\langle h, \xi \rangle\rangle$  in  $\mathcal{K}$  such that

$$(2.1) \quad \langle \langle\langle h, \xi \rangle\rangle, k \rangle = \langle \xi, h \otimes k \rangle, \forall k \in \mathcal{K}.$$

In other words,  $\langle\langle h, \xi \rangle\rangle = F_h^* \xi$ , where  $F_h \in \mathcal{B}(\mathcal{K}, \mathcal{H} \otimes \mathcal{K})$  is given by  $F_h k = h \otimes k$ .

Let  $\mathbf{h}$  and  $\mathcal{H}$  be two Hilbert spaces with orthonormal bases  $\{e_j : j \geq 1\}$  and  $\{\zeta_j : j \geq 1\}$ , respectively. For each  $A \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$  and  $u, v \in \mathbf{h}$  we define a linear operator  $A(u, v) \in \mathcal{B}(\mathcal{H})$  by

$$\langle \xi_1, A(u, v) \xi_2 \rangle = \langle u \otimes \xi_1, A v \otimes \xi_2 \rangle, \forall \xi_1, \xi_2 \in \mathcal{H}$$

and read off the following properties (for the proof, see Lemma 2.1 in [16]):

**Lemma 2.1.** *Let  $A, B \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$ . Then for any  $u, v, u_i, v_i \in \mathbf{h}$  ( $i = 1, 2$ ) we have*

- (i)  $A(\cdot, \cdot) : \mathbf{h} \times \mathbf{h} \mapsto \mathcal{B}(\mathcal{H})$  is a separately continuous sesqui-linear map, and if  $A(u, v) = B(u, v)$  for all  $u, v \in \mathbf{h}$ , then  $A = B$ ,
- (ii)  $\|A(u, v)\| \leq \|A\| \|u\| \|v\|$  and  $A(u, v)^* = A^*(v, u)$ ,
- (iii)  $A(u_1, v_1)B(u_2, v_2) = [A(|v_1\rangle\langle u_2| \otimes 1_{\mathcal{H}})B](u_1, v_2)$ ,
- (iv)  $AB(u, v) = \sum_{j \geq 1} A(u, e_j)B(e_j, v)$ , where the series converges strongly,
- (v)  $0 \leq A(u, v)^*A(u, v) \leq \|u\|^2 A^*A(v, v)$ ,
- (vi) for any  $\xi_1, \xi_2 \in \mathcal{H}$  we have

$$\begin{aligned} \langle A(u_1, v_1)\xi_1, B(u_2, v_2)\xi_2 \rangle &= \sum_{j \geq 1} \langle u_2 \otimes \zeta_j, [B(|v_2\rangle\langle v_1| \otimes |\xi_2\rangle\langle \xi_1|)A^*] u_1 \otimes \zeta_j \rangle \\ &= \langle v_1 \otimes \xi_1, [A^*(|u_1\rangle\langle u_2| \otimes 1_{\mathcal{H}})B] v_2 \otimes \xi_2 \rangle. \end{aligned}$$

For each  $A \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$  and  $\epsilon \in \mathbb{Z}_2 = \{0, 1\}$ , we define an operator  $A^{(\epsilon)} \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$  by

$$A^{(\epsilon)} := \begin{cases} A & \text{if } \epsilon = 0, \\ A^* & \text{if } \epsilon = 1. \end{cases}$$

For  $1 \leq k \leq n$ , we define a unitary exchanging map  $P_{k,n} : \mathbf{h}^{\otimes n} \otimes \mathcal{H} \rightarrow \mathbf{h}^{\otimes n} \otimes \mathcal{H}$  by

$$P_{k,n}(u_1 \otimes \cdots \otimes u_n \otimes \xi) := u_{\tau_{k,n}(1)} \otimes \cdots \otimes u_{\tau_{k,n}(n)} \otimes \xi$$

on product vectors, where  $\tau_{k,n} := (k \ k+1 \ \cdots \ n)$  is a permutation on  $\{1, 2, \dots, n\}$ . Let  $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \mathbb{Z}_2^n$ . Consider the ampliation of the operator  $A^{(\epsilon_k)}$  in  $\mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$  given by

$$A^{(n, \epsilon_k)} := P_{k,n}^*(1_{\mathbf{h}^{\otimes n-1}} \otimes A^{(\epsilon_k)})P_{k,n}.$$

Now we define the operator

$$A^{(\underline{\epsilon})} := \prod_{k=1}^n A^{(n, \epsilon_k)} := A^{(n, \epsilon_1)} \cdots A^{(n, \epsilon_n)}$$

as in  $\mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$ . Note that as here, through out this article, the product symbol  $\prod_{k=1}^n$  stands for product with the ordering from 1 to  $n$ . For product vectors  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$  one can see that

$$(2.2) \quad A^{(\underline{\epsilon})}(\underline{u}, \underline{v}) = \left( \prod_{i=1}^n A^{(n, \epsilon_i)} \right) (\underline{u}, \underline{v}) = \prod_{i=1}^n A^{(\epsilon_i)}(u_i, v_i) \in \mathcal{B}(\mathcal{H}),$$

moreover, for  $1 \leq m \leq n$ , we see that

$$(2.3) \quad \left( \prod_{i=1}^m A^{(n, \epsilon_i)} \right) (\underline{u}, \underline{v}) = \prod_{i=1}^m A^{(\epsilon_i)}(u_i, v_i) \prod_{i=m+1}^n \langle u_i, v_i \rangle \in \mathcal{B}(\mathcal{H}).$$

When  $\underline{\epsilon} = \underline{0} \in \mathbb{Z}_2^n$ , for simplicity we shall write  $A^{(n, k)}$  for  $A^{(n, \epsilon_k)}$  and  $A^{(n)}$  for  $A^{(\underline{\epsilon})}$ .

### 3. Unitary Processes with Independent Increments

Let  $\{U_{s,t} : 0 \leq s \leq t < \infty\}$  be a family of unitary operators in  $\mathcal{B}(\mathbf{h} \otimes \mathcal{H})$  with  $U_{s,s} = 1$  for any  $s \geq 0$  and  $\Omega$  be a fixed unit vector in  $\mathcal{H}$ . Let us consider the family of unitary operators  $\{U_{s,t}^{(\epsilon)}\}$  in  $\mathcal{B}(\mathbf{h} \otimes \mathcal{H})$  for  $\epsilon \in \mathbb{Z}_2$  given by  $U_{s,t}^{(0)} = U_{s,t}$  and  $U_{s,t}^{(1)} = U_{s,t}^*$ . As in Section 2, for fixed  $n \geq 1$ ,  $\underline{\epsilon} \in \mathbb{Z}_2^n$  and each  $1 \leq k \leq n$ , we define the families of operators  $\{U_{s,t}^{(n, \epsilon_k)}\}$  and  $\{U_{s,t}^{(\underline{\epsilon})}\}$  in  $\mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$ . By identity (2.2), for product vectors  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$  and  $\underline{\epsilon} \in \mathbb{Z}_2^n$ , we have

$$U_{s,t}^{(\underline{\epsilon})}(\underline{u}, \underline{v}) = \prod_{i=1}^n U_{s,t}^{(\epsilon_i)}(u_i, v_i).$$

Furthermore, for  $\underline{s} = (s_1, s_2, \dots, s_n)$ ,  $\underline{t} = (t_1, t_2, \dots, t_n)$  such that  $0 \leq s_1 \leq t_1 \leq s_2 \leq \dots \leq s_n \leq t_n < \infty$ , we define  $U_{\underline{s}, \underline{t}}^{(\underline{\epsilon})} \in \mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$  by setting

$$(3.1) \quad U_{\underline{s}, \underline{t}}^{(\underline{\epsilon})} := \prod_{k=1}^n U_{s_k, t_k}^{(n, \epsilon_k)}.$$

Then for  $\underline{u} = \otimes_{k=1}^n u_k, \underline{v} = \otimes_{k=1}^n v_k \in \mathbf{h}^{\otimes n}$  we have

$$U_{\underline{s}, \underline{t}}^{(\underline{\epsilon})}(\underline{u}, \underline{v}) = \prod_{k=1}^n U_{s_k, t_k}^{(\epsilon_k)}(u_k, v_k).$$

When  $\underline{\epsilon} = \underline{0}$ , we write  $U_{\underline{s}, \underline{t}}$  for  $U_{\underline{s}, \underline{t}}^{(\underline{\epsilon})}$ . For  $\alpha, \beta \geq 0$  and  $\underline{s} = (s_1, s_2, \dots, s_n), \underline{t} = (t_1, t_2, \dots, t_n)$ , we write  $\alpha \leq \underline{s}, \underline{t} \leq \beta$  if  $\alpha \leq s_1 \leq t_1 \leq s_2 \leq \dots \leq s_n \leq t_n \leq \beta$ .

We assume the following on the family of unitary  $\{U_{s,t} \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})\}$ .

**Assumption A:**

- (A1) (**Evolution**) for any  $0 \leq r \leq s \leq t < \infty$ ,  $U_{r,s}U_{s,t} = U_{r,t}$  and  $U_{s,s} = 1$ ,
- (A2) (**Independence of increments**) for any  $0 \leq s_i \leq t_i < \infty$  ( $i = 1, 2$ ) such that  $[s_1, t_1] \cap [s_2, t_2] = \emptyset$ ,
  - (i)  $U_{s_1, t_1}(u_1, v_1)$  commutes with  $U_{s_2, t_2}(u_2, v_2)$  and  $U_{s_2, t_2}^*(u_2, v_2)$  for any  $u_i, v_i \in \mathbf{h}$  ( $i = 1, 2$ ).
  - (ii) for any  $s_1 \leq \underline{q}, \underline{r} \leq t_1, s_2 \leq \underline{s}, \underline{t} \leq t_2$  and  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}, \underline{w}, \underline{z} \in \mathbf{h}^{\otimes m}$  and  $\underline{\epsilon} \in \mathbb{Z}_2^n, \underline{\epsilon}' \in \mathbb{Z}_2^m$ ,

$$\langle \Omega, U_{\underline{q}, \underline{r}}^{(\underline{\epsilon})}(\underline{u}, \underline{v}) U_{\underline{s}, \underline{t}}^{(\underline{\epsilon}')}(\underline{w}, \underline{z}) \Omega \rangle = \langle \Omega, U_{\underline{q}, \underline{r}}^{(\underline{\epsilon})}(\underline{u}, \underline{v}) \Omega \rangle \langle \Omega, U_{\underline{s}, \underline{t}}^{(\underline{\epsilon}')}(\underline{w}, \underline{z}) \Omega \rangle.$$

**Assumption B: (Regularity)** for any  $\infty > t \geq s \geq 0$ ,

$$\sup \{ |\langle \Omega, (U_{s,t} - 1)(u, v) \Omega \rangle| : \|u\| = \|v\| = 1 \} \leq C|t - s|$$

for some positive constant  $C$  independent of  $s, t$ .

*Remark 3.1.* Unlike [16, 17], in the **Assumption A**, the stationarity condition is not assumed.

As in [16, 17], we need further assumptions for Gaussianity and minimality:

**Assumption C: (Gaussianity)** for each  $t \geq s \geq 0$  and any  $u_k, v_k \in \mathbf{h}, \epsilon_k \in \mathbb{Z}_2$  ( $k = 1, 2, 3$ ),

$$(3.2) \quad \lim_{t \downarrow s} \frac{1}{t - s} \left\langle \Omega, \left( \prod_{k=1}^3 (U_{s,t}^{(\epsilon_k)} - 1)(u_k, v_k) \right) \Omega \right\rangle = 0.$$

**Assumption D: (Minimality)** the set

$$\mathcal{S}_0 = \left\{ U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v}) \Omega : \begin{array}{l} \underline{s} = (s_1, s_2, \dots, s_n), \underline{t} = (t_1, t_2, \dots, t_n) \text{ with } 0 \leq \underline{s}, \underline{t} < \infty, \\ \underline{u} = \otimes_{k=1}^n u_k, \underline{v} = \otimes_{k=1}^n v_k \in \mathbf{h}, n \geq 1 \end{array} \right\}$$

is total in  $\mathcal{H}$ .

*Remark 3.2.* The **Assumption D** is not really a restriction, one can as well work by replacing  $\mathcal{H}$  by the closure of the linear span of  $\mathcal{S}_0$ .

**3.1. Vacuum Expectation.** Let us look at the various evolutions associated with the  $\{U_{s,t}\}$ . Define a two parameter family of operators  $\{T_{s,t}\}$  on  $\mathbf{h}$  by

$$\langle u, T_{s,t}v \rangle := \langle \Omega, U_{s,t}(u, v)\Omega \rangle, \quad \forall u, v \in \mathbf{h}.$$

For each  $t \geq s \geq 0$ , since  $U_{s,t}$  is unitary,  $T_{s,t}$  is a contractions.

*Remark 3.1.* The **Assumption B** implies  $\|T_{s,t} - 1\| \leq C|t - s|$ . In particular  $\lim_{t \downarrow s} T_{s,t} = 1$  uniformly in  $s$ .

**Lemma 3.3.** *Under the **Assumptions A** and **B**, the family  $\{T_{s,t}\}$  of contractions satisfies*

- (i) *for any  $r \leq s \leq t < \infty$ ,  $T_{r,s}T_{s,t} = T_{r,t}$  and  $T_{s,s} = 1_{\mathbf{h}}$*
- (ii) *for any  $t' \geq t \geq s \geq 0$ ,  $\|T_{s,t'} - T_{s,t}\| \leq C|t' - t|$ .*

*Proof.* (i) The evolution and independent increment property of  $\{U_{s,t}\}$  and the definition of  $T_{s,t}$  gives the result.

(ii) By (i), for a fixed  $s \geq 0$  and any  $t' \geq t \geq s$ , we have

$$\|T_{s,t'} - T_{s,t}\| = \|T_{s,t}(T_{t,t'} - 1)\| \leq \|T_{s,t}\| \|T_{t,t'} - 1\| \leq C|t' - t|.$$

Therefore, by **Assumption B**, we have  $\lim_{t' \downarrow t} \|T_{s,t'} - T_{s,t}\| = 0$ .  $\square$

Let us note down the following result about the generator of the evolutions of the type  $T_{s,t}$  which is proved in the Appendix.

*Theorem 3.2.* There exists a Lebesgue measurable function  $G : \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbf{h})$  such that  $G$  is locally essentially bounded and

$$T_{s,t} - 1 = \int_s^t T_{s,\tau} G(\tau) d\tau.$$

$$(3.3) \quad \lim_{h \downarrow 0} \frac{T_{t,t+h} - I}{h} = G(t)$$

in the operator norm topology for almost every  $t$ .

We shall need the following observation (see Equation (6.2) in [16]):

$$(3.4) \quad \sum_{k \geq 1} \|(U_{s,t} - 1)(\phi_k, w)\Omega\|^2 = \langle w, (1 - T_{s,t})w \rangle + \langle (1 - T_{s,t})w, w \rangle$$

for any  $w \in \mathbf{h}$ , where  $\{\phi_k\}$  is an complete orthonormal basis of  $\mathbf{h}$ .

**Lemma 3.4.** *Under the **Assumptions C**, for each  $s \geq 0$ , we have the following:*

- (i) *for any  $n \geq 3$ ,  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$  and  $\underline{\epsilon} \in \mathbb{Z}_2^n$ ,*

$$(3.5) \quad \lim_{t \downarrow s} \frac{1}{t - s} \left\langle \Omega, \left( \prod_{k=1}^n \left[ (U_{s,t}^{(\epsilon_k)} - 1)(u_k, v_k) \right] \right) \Omega \right\rangle = 0,$$

- (ii) *for any vectors  $u, v \in \mathbf{h}$ , product vectors  $\underline{u}, \underline{z} \in \mathbf{h}^{\otimes n}$  and  $\epsilon \in \mathbb{Z}_2$ ,  $\underline{\epsilon}' \in \mathbb{Z}_2^n$ ,*

$$(3.6) \quad \begin{aligned} & \lim_{t \downarrow s} \frac{1}{t - s} \left\langle (U_{s,t} - 1)^{(\epsilon)}(u, v)\Omega, \left( U_{s,t}^{(\underline{\epsilon}')} - 1 \right)(\underline{u}, \underline{z})\Omega \right\rangle \\ &= (-1)^\epsilon \lim_{t \downarrow s} \frac{1}{t - s} \left\langle (U_{s,t} - 1)(u, v)\Omega, \left( U_{s,t}^{(\underline{\epsilon}')} - 1 \right)(\underline{u}, \underline{z})\Omega \right\rangle. \end{aligned}$$

*Proof.* (i) The proof is a simple modification of the proof of Lemma 6.6 in [16].

(ii) The idea here is similar to that in the proof of Lemma 6.7 in [16]. For  $\epsilon = 0$ , it is obvious. To see this for  $\epsilon = 1$ , put

$$\Phi = \left( U_{s,t}^{(\epsilon')} - 1 \right) (\underline{\mathbf{p}}, \underline{\mathbf{w}})$$

and consider the following

$$\begin{aligned} (3.7) \quad & \lim_{t \downarrow s} \frac{1}{t-s} \langle (U_{s,t} + U_{s,t}^* - 2) (u, v) \Omega, \Phi \Omega \rangle \\ &= - \lim_{t \downarrow s} \frac{1}{t-s} \langle [(U_{s,t}^* - 1) (U_{s,t} - 1)] (u, v) \Omega, \Phi \Omega \rangle \\ &= - \lim_{t \downarrow s} \frac{1}{t-s} \sum_{k \geq 1} \langle (U_{s,t} - 1) (e_k, v) \Omega, (U_{s,t} - 1) (e_k, u) \Phi \Omega \rangle. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \left| \frac{1}{t-s} \sum_{k \geq 1} \langle (U_{s,t} - 1) (e_k, v) \Omega, (U_{s,t} - 1) (e_k, u) \Phi \Omega \rangle \right|^2 \\ & \leq \left( \sum_{k \geq 1} \frac{1}{t-s} \|(U_{s,t} - 1) (e_k, v) \Omega\|^2 \right) \left( \sum_{k \geq 1} \frac{1}{t-s} \|(U_{s,t} - 1) (e_k, u) \Phi \Omega\|^2 \right). \end{aligned}$$

By (3.4) and (iv) in Lemma, the above quantity is equal to

$$\begin{aligned} & 2Re \left\langle v, \frac{1 - T_{s,t}}{t-s} v \right\rangle \frac{1}{t-s} \langle \Phi \Omega, [(U_{s,t}^* - 1) (U_{s,t} - 1)] (u, u) \Phi \Omega \rangle \\ &= 2Re \left\langle v, \frac{1 - T_{s,t}}{t-s} v \right\rangle \frac{1}{t-s} \langle \Phi \Omega, (2 - U_{s,t}^* - U_{s,t}) (u, u) \Phi \Omega \rangle, \end{aligned}$$

and by (3.3),  $\lim_{t \downarrow s} \left\langle v, \frac{1 - T_{s,t}}{t-s} v \right\rangle = \langle v, G(s)v \rangle$  for any  $v \in \mathbf{h}$ . Also, by **Assumption C** we have

$$\lim_{t \downarrow s} \frac{1}{t-s} \langle \Phi \Omega, (2 - U_{s,t}^* - U_{s,t}) (u, u) \Phi \Omega \rangle = 0.$$

Therefore, we have

$$\lim_{t \downarrow s} \frac{1}{t-s} \sum_{k \geq 1} \left\langle (U_{s,t} - 1) (e_k, u) \Omega, (U_{s,t} - 1) (e_k, v) \left( U_{s,t}^{(\epsilon')} - 1 \right) (\underline{\mathbf{p}}, \underline{\mathbf{w}}) \Omega \right\rangle = 0,$$

which, by applying (3.7), implies (3.6).  $\square$

For each  $s \geq 0$  and for vectors  $u, v, p, w \in \mathbf{h}$  the identity (3.6) gives

$$\begin{aligned} (3.8) \quad & \lim_{t \downarrow s} \frac{1}{t-s} \left\langle (U_{s,t} - 1)^{(\epsilon)} (u, v) \Omega, (U_{s,t} - 1)^{(\epsilon')} (p, w) \Omega \right\rangle \\ &= (-1)^{\epsilon + \epsilon'} \lim_{t \downarrow s} \frac{1}{t-s} \langle (U_{s,t} - 1) (u, v) \Omega, (U_{s,t} - 1) (p, w) \Omega \rangle. \end{aligned}$$

We now introduce the partial trace  $\text{Tr}_{\mathcal{H}}$  which is a linear map from  $\mathcal{B}_1(\mathbf{h} \otimes \mathcal{H})$  to  $\mathcal{B}_1(\mathbf{h})$  defined by

$$\langle u, \text{Tr}_{\mathcal{H}}(B)v \rangle := \sum_{j \geq 1} \langle u \otimes \zeta_j, Bv \otimes \zeta_j \rangle, \quad \forall u, v \in \mathbf{h}$$

for  $B \in \mathcal{B}_1(\mathbf{h} \otimes \mathcal{H})$ . In particular,  $\text{Tr}_{\mathcal{H}}(B) = \text{Tr}(B_2) B_1$  for  $B = B_1 \otimes B_2$ . Then we define a family of operators  $\{Z_{s,t}\}_{0 \leq s \leq t}$  on the Banach space  $\mathcal{B}_1(\mathbf{h})$  by

$$(3.9) \quad Z_{s,t}(\rho) = \text{Tr}_{\mathcal{H}} [U_{s,t}(\rho \otimes |\Omega\rangle\langle\Omega|) U_{s,t}^*], \quad \rho \in \mathcal{B}_1(\mathbf{h}).$$

Thus, for any  $u, v, p, w \in \mathbf{h}$ , we have

$$(3.10) \quad \langle p, Z_{s,t}(|w\rangle\langle v|)u \rangle := \langle U_{s,t}(u, v)\Omega, U_{s,t}(p, w)\Omega \rangle.$$

For  $\rho \in \mathcal{B}_1(\mathbf{h})$ , by the definition of  $Z_{s,t}$  and trace norm (see page no. 47 in [3]), we have

$$\begin{aligned} \|Z_{s,t}(\rho)\|_1 &= \|\text{Tr}_{\mathcal{H}}[U_{s,t}(\rho \otimes |\Omega\rangle\langle\Omega|) U_{s,t}^*]\|_1 \\ &= \sup_{\phi, \psi: \text{ons of } \mathbf{h}} \sum_{k \geq 1} |\langle \phi_k, \text{Tr}_{\mathcal{H}} [U_{s,t}(\rho \otimes |\Omega\rangle\langle\Omega|) U_{s,t}^*] \psi_k \rangle| \\ &\leq \sup_{\phi, \psi: \text{ons of } \mathbf{h}} \sum_{j, k \geq 1} |\langle \phi_k \otimes \zeta_j, U_{s,t}(\rho \otimes |\Omega\rangle\langle\Omega|) U_{s,t}^* \psi_k \otimes \zeta_j \rangle| \\ &\leq \|U_{s,t}(\rho \otimes |\Omega\rangle\langle\Omega|) U_{s,t}^*\|_1 \leq \|\rho\|_1. \end{aligned}$$

Thus  $Z_{s,t}$  is contractive. For any  $u, v \in \mathbf{h}$ ,  $\|U_{s,t}(u, v)\Omega\|^2 = \langle u, Z_{s,t}(|v\rangle\langle v|)u \rangle$  and positivity of  $Z_{s,t}$  is clear.

**Lemma 3.5.** *Under the **Assumptions A** and **B**,  $\{Z_{s,t}\}$  is a family of positive contractive map on  $\mathcal{B}_1(\mathbf{h})$  satisfying*

- (i) for any  $0 \leq r \leq s \leq t < \infty$ ,  $Z_{r,s}Z_{s,t} = Z_{r,t}$ ,  $Z_{s,s} = 1$
- (ii) for any  $t' \geq t \geq s \geq 0$ ,  $\|Z_{s,t'} - Z_{s,t}\|_1 \leq 4C|t' - t|$ ,
- (iii) For any  $\rho \in \mathcal{B}_1(\mathbf{h})$ ,  $\text{Tr}(Z_{s,t}\rho) = \text{Tr}(\rho)$ .

*Proof.* (i) To prove evolution property of  $Z_{s,t}$  it is enough to show that for any  $u, v, p, w \in \mathbf{h}$ ,  $\langle U_{s,t}(u, v)\Omega, U_{s,t}(p, w)\Omega \rangle = \langle p, Z_{r,t}(|w\rangle\langle v|)u \rangle = \langle p, Z_{r,s}Z_{s,t}(|w\rangle\langle v|)u \rangle$ . This can be checked by using evolution and independent increment property of unitary family  $U_{s,t}$ .

(ii) For any rank one operator  $\rho = |w\rangle\langle v|$ ,  $w, v \in \mathbf{h}$ , we have

$$\begin{aligned}
\|(Z_{s,t} - 1)(|w \rangle \langle v|)\|_1 &= \sup_{\{\phi\}, \{\psi\} \text{ ons of } \mathbf{h}} \sum_{k \geq 1} |\langle \phi_k, (Z_{s,t} - 1)(|w \rangle \langle v|) \psi_k \rangle| \\
&= \sup_{\phi, \psi} \sum_{k \geq 1} |\langle U_{s,t}(\psi_k, v)\Omega, U_{s,t}(\phi_k, w)\Omega \rangle - \overline{\langle \psi_k, v \rangle} \langle \phi_k, w \rangle| \\
&\leq \sup_{\phi, \psi} \sum_{k \geq 1} |\langle (U_{s,t} - 1)(\psi_k, v)\Omega, (U_{s,t} - 1)(\phi_k, w)\Omega \rangle| \\
&\quad + \sup_{\phi, \psi} \sum_{k \geq 1} |\overline{\langle \psi_k, v \rangle} \langle \Omega, (U_{s,t} - 1)(\phi_k, w)\Omega| \\
&\quad + \sup_{\phi, \psi} \sum_{k \geq 1} |\overline{\langle \Omega, (U_{s,t} - 1)(\psi_k, v)\Omega \rangle} \langle \phi_k, w \rangle| \\
&\leq \sup_{\phi, \psi} \left[ \sum_{k \geq 1} \|(U_{s,t} - 1)(\psi_k, v)\Omega\|^2 \right]^{1/2} \left[ \sum_{k \geq 1} \|(U_{s,t} - 1)(\phi_k, w)\Omega\|^2 \right]^{1/2} \\
&\quad + \sup_{\phi, \psi} \left[ \sum_{k \geq 1} |\langle \psi_k, v \rangle|^2 \right]^{1/2} \left[ \sum_{k \geq 1} |\langle \phi_k, (T_{s,t} - 1)w \rangle|^2 \right]^{1/2} \\
&\quad + \sup_{\phi, \psi} \left[ \sum_{k \geq 1} |\langle \phi_k, w \rangle|^2 \right]^{1/2} \left[ \sum_{k \geq 1} |\langle \psi_k, (T_{s,t} - 1)v \rangle|^2 \right]^{1/2}.
\end{aligned}$$

Hence by identity (3.4) and **Assumption B** we obtain

$$\begin{aligned}
&\|(Z_{s,t} - 1)(|w \rangle \langle v|)\|_1 \\
&\leq 2\|(T_{s,t} - 1)\| \|w\| \|v\| + \|(T_{s,t} - 1)w\| \|v\| + \|(T_{s,t} - 1)v\| \|w\| \\
&\leq 4C|t - s| \|w\| \|v\|.
\end{aligned}$$

Now any for  $\rho = \sum_k \lambda_k |\phi_k \rangle \langle \psi_k| \in \mathcal{B}_1(\mathbf{h})$ , where  $\{\phi_k\}$  and  $\{\psi_k\}$  are two orthonormal bases of  $\mathbf{h}$  and we have

$$\|Z_{s,t}(\rho) - \rho\|_1 \leq 4C \left( \sum_k |\lambda_k| \right) |t - s| \leq 4C \|\rho\|_1 |t - s|$$

and hence

$$(3.11) \quad \|Z_{s,t} - 1\|_1 \leq 4C|t - s|.$$

By evolution property and contractivity of  $\{Z_{s,t}\}$

$$\|Z_{s,t'} - Z_{s,t}\| = \|Z_{s,t}(Z_{t,t'} - 1)\| \leq \|Z_{s,t}\| \|Z_{t,t'} - 1\| \leq 4C|t' - t|.$$

(iii) It can be proved as in lemma 6.5 in [16] □

The theorem 6.1 in the Appendix leads to following result for  $Z_{s,t}$ .

**Theorem 3.3.** Under the **Assumption A, B** there exists a Lebesgue measurable function  $\mathcal{L} : \mathbb{R}_+ \rightarrow \mathcal{B}(\mathcal{B}_1(\mathbf{h}))$  such that  $\mathcal{L}$  is locally essentially bounded in  $\mathbb{R}_+$  and such that

$$Z_{s,t} - 1 = \int_s^t Z_{s,\tau} \mathcal{L}(\tau) d\tau, \quad \lim_{h \downarrow 0} \frac{Z_{t,t+h} - I}{h} = \mathcal{L}(t).$$



#### 4. Construction of Noise Space

Put  $M_0 := \{(\underline{u}, \underline{v}, \underline{\epsilon}) : \underline{u} = \otimes_{i=1}^n u_i, \underline{v} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}, \underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{Z}_2^n, n \geq 1\}$ . Now, consider the relation “ $\sim$ ” on  $M_0$  as defined in [16] :  $(\underline{u}, \underline{v}, \underline{\epsilon}) \sim (\underline{p}, \underline{w}, \underline{\epsilon}')$  if  $\underline{\epsilon} = \underline{\epsilon}'$  and  $|\underline{u} \rangle \langle \underline{v}| = |\underline{w} \rangle \langle \underline{z}| \in \mathcal{B}(\mathbf{h}^{\otimes n})$ . Now consider the algebra  $M$  generated by  $M_0 / \sim$  with multiplication structure given by  $(\underline{u}, \underline{v}, \underline{\epsilon}) \cdot (\underline{p}, \underline{w}, \underline{\epsilon}') = (\underline{u} \otimes \underline{p}, \underline{v} \otimes \underline{w}, \underline{\epsilon} \oplus \underline{\epsilon}')$ . For each  $s \geq 0$  we define a scalar valued map  $K_s$  on  $M \times M$  by setting, for  $(\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}') \in M_0$ ,

$$K_s((\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}')) := \lim_{t \downarrow s} \frac{1}{t-s} \left\langle \left( U_{s,t}^{(\underline{\epsilon})} - 1 \right) (\underline{u}, \underline{v}) \Omega, \left( U_{s,t}^{(\underline{\epsilon}')} - 1 \right) (\underline{p}, \underline{w}) \Omega \right\rangle$$

if the limit exists.

**Theorem 4.1.** *For almost every  $s$*

- (i) *the map  $K_s$  is a positive definite kernel on  $M$ ,*
- (ii) *there exists a unique (up to unitary equivalence) separable Hilbert space  $\mathbf{k}_s$ , an embedding  $\eta_s : M \rightarrow \mathbf{k}_s$  such that*

$$(4.1) \quad \{\eta_s(\underline{u}, \underline{v}, \underline{\epsilon}) : (\underline{u}, \underline{v}, \underline{\epsilon}) \in M_0\} \text{ is total in } \mathbf{k}_s,$$

$$(4.2) \quad \langle \eta_s(\underline{u}, \underline{v}, \underline{\epsilon}), \eta_s(\underline{p}, \underline{w}, \underline{\epsilon}') \rangle = K_s((\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}')),$$

- (iii) *for any  $(\underline{u}, \underline{v}, \underline{\epsilon}) \in M_0$ ,  $\underline{u} = \otimes_{i=1}^n u_i$ ,  $\underline{v} = \otimes_{i=1}^n v_i$  and  $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$*

$$(4.3) \quad \eta_s(\underline{u}, \underline{v}, \underline{\epsilon}) = \sum_{i=1}^n \prod_{k \neq i} \langle u_k, v_k \rangle \eta_s(u_i, v_i, \epsilon_i),$$

- (iv)  *$\eta_s(u, v, 1) = -\eta_s(u, v, 0)$  for any  $u, v \in \mathbf{h}$ ,*
- (v) *for fixed  $u, v, p, w \in \mathbf{h}$ , the map  $s \mapsto K_s((u, v), (p, w)) = \langle \eta_s(u, v), \eta_s(p, w) \rangle$  is Lebesgue measurable and essentially bounded in  $\mathbb{R}_+$ .*

*Proof.* (i) The proof is exactly same as the proof of Lemma 7.1 in [16]. By Lemma 3.4, for elements  $(\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}') \in M_0$ ,  $\underline{\epsilon} \in \mathbb{Z}_2^m$  and  $\underline{\epsilon}' \in \mathbb{Z}_2^n$ , we have

$$(4.4) \quad \begin{aligned} & K_s((\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}')) \\ &= \lim_{t \downarrow s} \frac{1}{t-s} \left\langle \left( U_{s,t}^{(\underline{\epsilon})} - 1 \right) (\underline{u}, \underline{v}) \Omega, \left( U_{s,t}^{(\underline{\epsilon}')} - 1 \right) (\underline{p}, \underline{w}) \Omega \right\rangle \\ &= \sum_{1 \leq i \leq m, 1 \leq j \leq n} \prod_{k \neq i} \overline{\langle u_k, v_k \rangle} \prod_{l \neq j} \langle p_l, w_l \rangle \\ &\quad \times \lim_{t \downarrow s} \frac{1}{t-s} \left\langle (U_{s,t} - 1)^{(\epsilon_i)} (u_i, v_i) \Omega, (U_{s,t} - 1)^{(\epsilon'_j)} (p_j, w_j) \Omega \right\rangle. \end{aligned}$$

Since

$$\begin{aligned} & \langle (U_{s,t} - 1) (u, v) \Omega, (U_{s,t} - 1) (p, w) \Omega \rangle \\ &= \langle U_{s,t}(u, v) \Omega, U_{s,t}(p, w) \Omega \rangle - \overline{\langle u, v \rangle} \langle p, w \rangle \\ &\quad - \overline{\langle u, v \rangle} \langle \Omega, (U_{s,t} - 1) (p, w) \Omega \rangle - \overline{\langle \Omega, (U_{s,t} - 1) (u, v) \Omega \rangle} \langle p, w \rangle \\ &= \langle p, (Z_{s,t} - 1) (|w \rangle \langle v|) u \rangle - \overline{\langle u, v \rangle} \langle p, (T_{s,t} - 1) w \rangle - \overline{\langle u, (T_{s,t} - 1) v \rangle} \langle p, w \rangle, \end{aligned}$$

the existence of the limits on the right hand side of (4.4) follows from the identity (3.6) and theorems 3.2 and 3.3 and  $K_s$  is given by

$$\begin{aligned}
(4.5) \quad K_s((u, v, \epsilon), (p, w, \epsilon')) &= (-1)^{\epsilon+\epsilon'} \lim_{t \downarrow s} \left\{ \left\langle p, \frac{Z_{s,t}-1}{t-s} (|w \succ v|)u \right\rangle - \overline{\langle u, v \rangle} \left\langle p, \frac{T_{s,t}-1}{t-s} w \right\rangle \right\} \\
&\quad - (-1)^{\epsilon+\epsilon'} \lim_{t \downarrow s} \left\langle u, \frac{T_{s,t}-1}{t-s} v \right\rangle \langle p, w \rangle \\
&= (-1)^{\epsilon+\epsilon'} \left\{ \langle p, \mathcal{L}(s)(|w \succ v|)u \rangle - \overline{\langle u, v \rangle} \langle p, G(s)w \rangle - \overline{\langle u, G(s)v \rangle} \langle p, w \rangle \right\}.
\end{aligned}$$

This expression can be extend to the algebra  $M$  by sesqui-linearly.

(ii) For each  $s \geq 0$ , the Kolmogorov's construction [14] to the pair  $(M, K_s)$  provides a Hilbert space  $\mathbf{k}_s$  as the closure of the span of  $\{\eta_s(\underline{u}, \underline{v}, \underline{\epsilon}) : (\underline{u}, \underline{v}, \underline{\epsilon}) \in M\}$ .

(iii) Again as in [16], for any  $(\underline{p}, \underline{w}, \underline{\epsilon}') \in M_0$ , by Lemma 3.4, we have

$$\begin{aligned}
\langle \eta_s(\underline{u}, \underline{v}, \underline{\epsilon}), \eta_s(\underline{p}, \underline{w}, \underline{\epsilon}') \rangle &= K_s((\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}')) \\
&= \sum_{i=1}^n \prod_{k \neq i} \overline{\langle u_k, v_k \rangle} \langle \eta_s(u_i, v_i, \epsilon_i), \eta_s(\underline{p}, \underline{w}, \underline{\epsilon}') \rangle.
\end{aligned}$$

Since  $\{\eta_s(\underline{p}, \underline{w}, \underline{\epsilon}') : (\underline{p}, \underline{w}, \underline{\epsilon}') \in M_0\}$  is a total subset of  $\mathbf{k}_s$ , (4.3) follows.

(iv) By (3.6), we have

$$\langle \eta_s(u, v, 1), \eta_s(\underline{p}, \underline{w}, \underline{\epsilon}') \rangle = \langle -\eta_s(u, v, 0), \eta_s(\underline{p}, \underline{w}, \underline{\epsilon}') \rangle$$

and hence  $\eta_s(u, v, 1) = -\eta_s(u, v, 0)$ .

By parts (iii) and (iv) of this theorem, it is clear that  $\mathbf{k}_s$  is spanned by the family  $\{\eta_s(u, v) : u, v \in \mathbf{h}\}$ , where we have written  $\eta_t(u, v)$  for  $\eta_t(u, v, 0)$ .

Since  $G(s), \mathcal{L}(s)$  are essentially bounded in norm it follow from (4.5) that  $\eta_s(., .) : \mathbf{h} \times \mathbf{h} \rightarrow \mathbf{k}_s$  is continuous and thus separability of  $\mathbf{k}_s$  follows from that of  $\mathbf{h}$ .

(v) Since  $\mathcal{L}(s)$  and  $G(s)$  are measurable essentially bounded, result follows from the identity (4.5).  $\square$

For any two orthonormal bases  $\{\phi_k\}, \{\psi_l\}$  of  $\mathbf{h}$ , the collection of vectors  $\{\eta_s(\phi_k, \psi_l) : k, l \geq 1\}$  is a countable total family in  $\mathbf{k}_s$  and  $s \mapsto \langle \eta_s(u, v), \eta_s(p, w) \rangle = K_s((u, v), (p, w))$  is a Lebesgue measurable function. Thus  $s \mapsto \langle \eta_s(u, v)$  is measurable. The family  $\{\mathbf{k}_s : s \geq 0\}$  spanned by  $\{\eta_s(u, v) : s \geq 0, u, v \in \mathbf{h}\}$ , is a measurable field of Hilbert spaces [5].

For any  $T \geq 0$ , define  $K^T((u, v), (p, w)) = \int_0^T K_s((u, v), (p, w)) ds$

$$= \int_0^T \left\{ \langle p, \mathcal{L}(s)(|w \succ v|)u \rangle - \overline{\langle u, v \rangle} \langle p, G(s)w \rangle - \overline{\langle u, G(s)v \rangle} \langle p, w \rangle \right\} ds.$$

Since each  $K_s$  is positive definite it can be seen that  $K^T$  is a positive definite kernel. Let the associated Hilbert space  $\mathbf{k}^T$ . There exists a family of vectors  $\eta^T(u, v)$  which spans the Hilbert space  $\mathbf{k}^T$  such that

$$\langle \eta^T(u, v), \eta^T(p, w) \rangle = K^T((u, v), (p, w))$$

$$= \int_0^T K_s((u, v), (p, w)) ds = \int_0^T \langle \eta_s(u, v), \eta_s(p, w) \rangle ds$$

In  $\mathbf{k}^T$  there exists a bounded self adjoint operator  $A$  with absolutely continuous simple spectrum such that  $A\eta^T(u, v)(s) = s\eta_s(u, v)$  for almost every  $s \in [0, T]$  and  $\mathbf{k}^T$  is the direct integral  $\int_{[0, T]}^\oplus \mathbf{k}_s ds$  (Ref [5]). There is natural isometric embedding of  $\mathbf{k}^T$  in  $\mathbf{k}^{T'}$  for  $T \leq T'$  by setting  $\eta_s^{T, T'}(u, v) = \eta_s^T(u, v)$  for all  $0 \leq s \leq T$  and 0 for  $s \in (T, T']$ .

*Remark 4.1.* The integral  $\int_{\mathbb{R}_+} K_s((u, v), (u, v)) ds = \int_{\mathbb{R}_+} \|\eta_s(u, v)\|^2 ds$  need not exist and therefore  $\int_{\mathbb{R}_+}^\oplus \mathbf{k}_s ds$  may not be defined.

**Lemma 4.2.** *Under the hypothesis of Theorem 4.1, we have the following:*

- (i) *There exists a unique strong measurable family of bounded operators  $L(t) : \mathbf{h} \rightarrow \mathbf{h} \otimes \mathbf{k}_t$  such that*

$$\|L(t)v\|^2 = -2\operatorname{Re} \langle v, G(t)v \rangle, \quad \forall v \in \mathbf{h}.$$

- (ii) *The map  $t \mapsto L(t)$  is essentially norm bounded.*

*Proof.* (i) By the identity (4.5), for any  $u, v \in \mathbf{h}$ , we have for almost every  $t \geq 0$

$$(4.6) \quad \|\eta_t(u, v)\|^2 = \langle u, \mathcal{L}(t)(|v\rangle\langle v|)u \rangle - \overline{\langle u, v \rangle} \langle u, G(t)v \rangle - \overline{\langle u, G(t)v \rangle} \langle u, v \rangle$$

and thus

$$\begin{aligned} \sum_k \|e_k \otimes \eta_t(e_k, v)\|^2 &= \sum_k \|\eta_t(e_k, v)\|^2 \\ &= \sum_k \left[ \langle e_k, \mathcal{L}(t)(|v\rangle\langle v|)e_k \rangle - \overline{\langle e_k, v \rangle} \langle e_k, G(t)v \rangle - \overline{\langle e_k, G(t)v \rangle} \langle e_k, v \rangle \right] \\ &= \operatorname{Tr}(\mathcal{L}(t)(|v\rangle\langle v|)) - \langle v, G(t)v \rangle - \overline{\langle v, G(t)v \rangle}. \end{aligned}$$

Moreover, since  $Z_{s,t}$  is trace preserving it follows that  $\operatorname{Tr}(\mathcal{L}(t)(|v\rangle\langle v|)) = 0$ . Therefore  $\sum_k \|e_k \otimes \eta_t(e_k, v)\|^2 = -2\operatorname{Re} \langle v, G(t)v \rangle$ . This implies that  $\sum_k e_k \otimes \eta_t(e_k, v)$  is convergent in norm and in fact for almost every  $t$  it defines a bounded operator  $L(t) : \mathbf{h} \rightarrow \mathbf{h} \otimes \mathbf{k}_t$  given by  $L(t)v = \sum_k e_k \otimes \eta_t(e_k, v)$  with

$$(4.7) \quad \|L(t)v\|^2 = -2\operatorname{Re} \langle v, G(t)v \rangle.$$

The strong measurability of  $t \mapsto L(t)$  follows from the definition.

The part (ii) follows from the essential norm boundedness of  $G(\cdot)$ . □

## 5. Hudson-Parthasarathy (HP) Evolution Systems and Equivalence

**5.1. HP Evolution Systems.** In order to simplify the discussion of the existence and uniqueness of the solution of HP type quantum stochastic differential equation in  $\Gamma_{\text{sym}}(\int_{\mathbb{R}_+}^\oplus \mathbf{k}_s ds)$  and to be able to refer to existing literature, it is convenient to introduce the following point of view which allow us to embed the process in the standard Fock space  $\Gamma = \Gamma_{\text{sym}}(L^2(\mathbb{R}_+, \mathbf{k}))$  where  $\mathbf{k} = l^2(\mathbb{N})$ .

Note that for almost every  $t \geq 0$ ,  $\mathbf{k}_t$  is a complex separable Hilbert space. Setting  $d(t) =$  the dimension of  $\mathbf{k}_t$ ,  $d : \mathbb{R}_+ \rightarrow \mathbb{N} \cup \{\infty\}$  is measurable and defining  $\Lambda_n = \{t : d(t) = n\}$ ,  $\mathbb{R}_+$  can be written as disjoint union  $\bigcup_{n=1}^\infty \Lambda_n$  of measurable sets. Let us consider the Hilbert space  $l^2(\mathbb{N})$  with a fixed orthonormal basis  $\{E_j : j \geq 0\}$ . Now for  $t \in \Lambda_n$ ,  $n < \infty$

we embed  $\mathbf{k}_t$  as the  $n$  dimensional subspace  $Span\{E_j : 1 \leq j \leq n\}$  of  $\mathbf{k}$  and for  $t \in \Lambda_\infty$ ,  $\mathbf{k}_t$  identified with  $\mathbf{k}$ . Then the direct integral  $\int_{\mathbb{R}_+}^\oplus \mathbf{k}_t dt = \bigoplus_{n \geq 1} L^2(\Lambda_n, \mathbb{C}^n) \subseteq L^2(\mathbb{R}_+, \mathbf{k})$ . If  $\Lambda_\infty = \emptyset$ , then  $\int_{\mathbb{R}_+}^\oplus \mathbf{k}_t dt$  is isomprphic to  $L^2(\mathbb{R}_+, \mathbb{C}^n)$  for some  $n$ .

For any subset  $\mathbf{D} \subseteq L^2(\mathbb{R}_+, \mathbf{k})$ , let  $\mathcal{E}(\mathbf{D})$  be the subspace of  $\Gamma$  which is spanned by the set  $\{\mathbf{e}(f) : f \in \mathbf{D}\}$  of exponential vectors defined as:

$$\mathbf{e}(f) := \bigoplus_{n \geq 0} \frac{f^{\otimes n}}{\sqrt{n!}}.$$

For  $0 \leq s < t < \infty$  and  $f \in \mathcal{K} = L^2(\mathbb{R}_+, \mathbf{k})$ , we denote the functions  $1_{[0,s]}f$ ,  $1_{(s,t]}f$  and  $1_{[t,\infty)}f$  by  $f_s$ ,  $f_{(s,t]}$  and  $f_t$ , where  $1_A$  is the indicator function of  $A \subset [0, \infty)$ . Then the Hilbert spaces  $\mathcal{K}$  and  $\Gamma$  can be decomposed as  $\mathcal{K} = \mathcal{K}_s \oplus \mathcal{K}_{(s,t]} \oplus \mathcal{K}_t$  and  $\Gamma = \Gamma_s \otimes \Gamma_{(s,t]} \otimes \Gamma_t$  via the unitary isomorphism given by:

$$\Gamma \ni \mathbf{e}(f) \longleftrightarrow \mathbf{e}(f_s) \otimes \mathbf{e}(f_{(s,t]}) \otimes \mathbf{e}(f_t) \in \Gamma_s \otimes \Gamma_{(s,t]} \otimes \Gamma_t,$$

where  $\mathcal{K}_s = L^2([0, s], \mathbf{k})$ ,  $\mathcal{K}_{(s,t]} = L^2((s, t], \mathbf{k})$ ,  $\mathcal{K}_t = L^2([t, \infty), \mathbf{k})$  and  $\Gamma_s = \Gamma(\mathcal{K}_s)$ ,  $\Gamma_{(s,t]} = \Gamma(\mathcal{K}_{(s,t]})$ ,  $\Gamma_t = \Gamma(\mathcal{K}_t)$ .

Let us consider the Hudson-Parthasarathy (HP) type equation on  $\mathbf{h} \otimes \Gamma$ :

$$(5.1) \quad V_{s,t} = 1_{\mathbf{h} \otimes \Gamma} + \sum_{\mu, \nu \geq 0} \int_s^t V_{s,\tau} L_\nu^\mu(\tau) \Lambda_\mu^\nu(d\tau).$$

Here the coefficients  $L_\nu^\mu(\tau)$  ( $\mu, \nu \geq 0$ ) are operators in  $\mathbf{h}$  and  $\Lambda_\mu^\nu(t)$  are fundamental processes define by

$$(5.2) \quad \Lambda_\mu^\nu(t) = \begin{cases} t 1_{\mathbf{h} \otimes \Gamma} & \text{for } (\mu, \nu) = (0, 0), \\ a(1_{[0,t]} \otimes E_j(t)) & \text{for } (\mu, \nu) = (j, 0), \\ a^\dagger(1_{[0,t]} \otimes E_k(t)) & \text{for } (\mu, \nu) = (0, k), \\ \Lambda(1_{[0,t]} \otimes |E_k(t) \rangle \langle E_j(t)|) & \text{for } (\mu, \nu) = (j, k), \end{cases}$$

where  $E_j(t) = E_j$  for  $j \in \{1, 2, \dots, d(t)\}$  and  $E_j(t) = 0$  otherwise. With respect to the orthonormal basis  $E_j(t)$  we have bounded operators  $\{L_j(t) : t \geq 0, j \geq 1\}$  in  $\mathbf{h}$  such that

$$(5.3) \quad \langle u, L_j(t)v \rangle = \langle E_j, \eta_t(u, v) \rangle, \forall u, v \in \mathbf{h}.$$

For detail about quantum stochastic calculus see [14, 6].

Now, let us state the main result of this article.

**Theorem 5.1.** *Under Assumptions **A**, **B**, **C** and **D**, we have the following.*

(i) *The HP type equation*

$$(5.4) \quad V_{s,t} = 1_{\mathbf{h} \otimes \Gamma} + \sum_{\mu, \nu \geq 0} \int_s^t V_{s,r} L_\nu^\mu(r) \Lambda_\mu^\nu(dr)$$

on  $\mathbf{h} \otimes \Gamma_{\text{sym}}(\mathcal{K})$  with coefficients  $L_\nu^\mu(t)$  given by

$$(5.5) \quad L_\nu^\mu(t) = \begin{cases} G(t) & \text{for } (\mu, \nu) = (0, 0), \\ L_j(t) & \text{for } (\mu, \nu) = (j, 0), \\ -L_k(t)^* & \text{for } (\mu, \nu) = (0, k), \\ 0 & \text{for } (\mu, \nu) = (j, k) \end{cases}$$

admit a unique unitary solution  $V_{s,t}$ .

(ii) *There exists a unitary isomorphism  $\tilde{\Xi} : \mathbf{h} \otimes \mathcal{H} \rightarrow \mathbf{h} \otimes \Gamma$  such that*

$$(5.6) \quad U_{s,t} = \tilde{\Xi}^* V_{s,t} \tilde{\Xi}, \quad \forall 0 \leq s \leq t < \infty.$$

Here we shall sketch the proof of part (i) of the theorem and postpone that of part (ii) to the next two sub sections. For  $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \mathbb{Z}_2^n$ , we define  $V_{s,t}^{(\underline{\epsilon})} \in \mathcal{B}(\mathbf{h}^{\otimes n} \otimes \Gamma)$  by setting  $V_{s,t}^{(\epsilon)} \in \mathcal{B}(\mathbf{h} \otimes \Gamma)$  by

$$V_{s,t}^{(\epsilon)} = \begin{cases} V_{s,t} & \text{for } \epsilon = 0, \\ V_{s,t}^* & \text{for } \epsilon = 1. \end{cases}$$

The next result verifies the properties of **Assumption A** for the family  $V_{s,t}$  with  $\Omega = \mathbf{e}(0) \in \Gamma$ .

**Lemma 5.2.** *The unitary solution  $\{V_{s,t}\}$  of HP equation (5.4) satisfies*

- (i) *for any  $0 \leq r \leq s \leq t < \infty$ ,  $V_{r,t} = V_{r,s} V_{s,t}$ ,*
- (ii) *for  $[q, r] \cap [s, t] = \emptyset$ ,  $V_{q,r}(u, v)$  commute with  $V_{s,t}(p, w)$  and  $V_{s,t}(p, w)^*$  for any  $u, v, p, w \in \mathbf{h}$ ,*
- (iii) *for any  $0 \leq s \leq t < \infty$ ,*

$$\langle \mathbf{e}(0), V_{s,t}(u, v) \mathbf{e}(0) \rangle = \langle u, T_{s,t} v \rangle, \quad \forall u, v \in \mathbf{h}.$$

*Proof.* (i) For fixed  $0 \leq r \leq s \leq t < \infty$ , we set  $W_{r,t} = V_{r,s} V_{s,t}$ . Then by (5.4), we have

$$\begin{aligned} W_{r,t} &= V_{r,s} + \sum_{\mu, \nu \geq 0} \int_s^t V_{r,s} V_{s,q} L_\nu^\mu(\tau) \Lambda_\mu^\nu(d\tau) \\ &= W_{r,s} + \sum_{\mu, \nu \geq 0} \int_s^t W_{r,q} L_\nu^\mu(\tau) \Lambda_\mu^\nu(d\tau), \end{aligned}$$

where  $W_{r,s} = V_{r,s} V_{s,s} = V_{r,s}$ . Thus the family  $\{W_{r,t}\}$  of unitary operators also satisfies the HP equation (5.4). Hence by uniqueness of the solution of this quantum stochastic differential equation,  $W_{r,t} = V_{r,t}$  for any  $0 \leq r \leq s \leq t < \infty$ , and the result follows.

(ii) For any  $0 \leq s \leq t < \infty$ ,  $V_{s,t} \in \mathcal{B}(\mathbf{h} \otimes \Gamma_{[s,t]})$ .  $p, w \in \mathbf{h}$ ,  $V_{s,t} p, w \in \mathcal{B}(\Gamma_{[s,t]})$  and the statement follows.

(iii) Let us define

$$\langle u, \tilde{T}_{s,t} v \rangle := \langle \mathbf{e}(0), V_{s,t}(u, v) \mathbf{e}(0) \rangle, \quad \forall u, v \in \mathbf{h}.$$

Then  $\tilde{T}_{s,t}$  is a contractive family of operators and by the cocycle property of  $V_{s,t}$ ,

$$(5.7) \quad \tilde{T}_{s,t} = 1 + \int_s^t \tilde{T}_{s,\tau} G(\tau) d\tau.$$

Thus  $\tilde{T}_{s,t} - T_{s,t}$  satisfies the differential equation

$$\tilde{T}_{s,t} - T_{s,t} = \int_s^t (\tilde{T}_{s,\tau} - T_{s,\tau}) G(\tau) d\tau.$$

Since  $G(\tau)$  is an essentially norm bounded function, an iteration of (5.7) will leads to  $\tilde{T}_{s,t} = T_{s,t}$  for all  $s, t$ .  $\square$

Consider the family of operators  $\tilde{Z}_{s,t}$  defined by

$$\tilde{Z}_{s,t}(\rho) = \text{Tr}_{\mathcal{H}} [V_{s,t}(\rho \otimes |\mathbf{e}(0)\rangle\langle \mathbf{e}(0)|)V_{s,t}^*], \quad \forall \rho \in \mathcal{B}_1(\mathbf{h}).$$

As for  $Z_{s,t}$ , it can be seen that  $\tilde{Z}_{s,t}$  is a contractive family of maps on  $\mathcal{B}_1(\mathbf{h})$  and, in particular, for any  $u, v, p, w \in \mathbf{h}$ ,

$$\langle p, \tilde{Z}_{s,t}(|w\rangle\langle v|)u \rangle = \langle V_{s,t}(u, v)\mathbf{e}(0), V_{s,t}(p, w)\mathbf{e}(0) \rangle.$$

**Lemma 5.3.** *The family  $\{\tilde{Z}_{s,t}\}$  is a uniformly continuous evolution of contraction on  $\mathcal{B}_1(\mathbf{h})$  and  $\tilde{Z}_{s,t} = Z_{s,t}$ , where  $Z_{s,t}$  is given as in (3.9).*

*Proof.* By (5.4) and Ito's formula, for  $u, v, p, w \in \mathbf{h}$

$$\begin{aligned} \langle p, [\tilde{Z}_{s,t} - 1] (|w\rangle\langle v|)u \rangle &= \langle V_{s,t}(u, v)\mathbf{e}(0), V_{s,t}(p, w)\mathbf{e}(0) \rangle - \overline{\langle u, v \rangle} \langle p, w \rangle \\ &= \int_s^t \langle V_{s,\tau}(u, v)\mathbf{e}(0), V_{s,\tau}(p, G(\tau)z)\mathbf{e}(0) \rangle d\tau + \int_s^t \langle V_{s,\tau}(u, G(\tau)v)\mathbf{e}(0), V_{s,\tau}(p, w)\mathbf{e}(0) \rangle d\tau \\ &\quad + \int_s^t \langle V_{s,\tau}(u, L_j(\tau)v)\mathbf{e}(0), V_{s,\tau}(p, L_j(\tau)z)\mathbf{e}(0) \rangle d\tau \\ &= \int_s^t \langle p, \tilde{Z}_{s,\tau}(|G(\tau)w\rangle\langle v|)u \rangle d\tau + \int_s^t \langle p, \tilde{Z}_{s,\tau}(|w\rangle\langle G(\tau)v|)u \rangle d\tau \\ &\quad + \sum_{j \geq 1} \int_s^t \langle p, \tilde{Z}_{s,\tau}(|L_j(\tau)w\rangle\langle L_j(\tau)v|)u \rangle d\tau. \end{aligned}$$

Thus by identity (5.3) for  $\{L_j(t)\}$ , we have that

$$(5.8) \quad \langle p, [\tilde{Z}_{s,t} - 1] (\rho)u \rangle = \int_s^t \langle p, \tilde{Z}_{s,\tau} \mathcal{L}(\tau)(\rho)u \rangle d\tau,$$

where  $\rho = |w\rangle\langle v|$ . Thus the family  $\{\tilde{Z}_{s,t}\}$  satisfies the differential equation

$$\tilde{Z}_{s,t}(\rho) = \rho + \int_s^t \tilde{Z}_{s,\tau} \mathcal{L}(\tau)(\rho) d\tau, \quad \rho \in \mathcal{B}_1(\mathbf{h}).$$

Therefore, proceeding as in the proof of Lemma 5.2 (iii) we can conclude that  $\tilde{Z}_{s,t} = Z_{s,t}$ .  $\square$

**5.2. Minimality of HP Evolution Systems.** In this section we shall show the minimality of the HP evolution system  $\{V_{s,t}\}$  discussed in Section 5.1 which will be needed to prove (ii) in Theorem 5.1, i.e., to establish unitary equivalence of  $U_{s,t}$  and  $V_{s,t}$ . We shall prove here that the subset

$$\mathcal{S}' = \left\{ V_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0) : \begin{array}{l} \underline{s} = (s_1, s_2, \dots, s_n), \underline{t} = (t_1, t_2, \dots, t_n) \text{ with } 0 \leq \underline{s}, \underline{t} < \infty, \\ \underline{u} = \otimes_{i=1}^n u_i, \underline{v} = \otimes_{i=1}^n v_i \in \mathbf{h}, n \geq 1 \end{array} \right\}$$

is total in the symmetric Fock space  $\Gamma(\mathcal{K}) \subseteq \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ , where

$$V_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0) := V_{s_1, t_1}(u_1, v_1) \cdots V_{s_n, t_n}(u_n, v_n)\mathbf{e}(0).$$

Let  $\tau \geq 0$  be fixed and as in (Ref. [16]), we note that for any  $0 \leq s < t \leq \tau$ ,  $u, v \in \mathbf{h}$ ,

$$(5.9) \quad \frac{1}{t-s} [V_{s,t} - 1] (u, v)\mathbf{e}(0) = \gamma(s, t, u, v) + \rho(s, t, u, v) + \zeta(s, t, u, v) + \varsigma(s, t, u, v),$$

where these vectors in the Fock space  $\Gamma$  are given by

$$\begin{aligned}\gamma(s, t, u, v) &:= \frac{1}{t-s} \sum_{j \geq 1} \int_s^t \langle u, L_j(\lambda) v \rangle a_j^\dagger(d\lambda) \mathbf{e}(0), \\ \rho(s, t, u, v) &:= \frac{1}{t-s} \int_s^t \langle u, G(\lambda) v \rangle d\lambda \mathbf{e}(0), \\ \zeta(s, t, u, v) &:= \frac{1}{t-s} \sum_{j \geq 1} \int_s^t (V_{s,\lambda} - 1) \langle u, L_j(\lambda) v \rangle a_j^\dagger(d\lambda) \mathbf{e}(0), \\ \varsigma(s, t, u, v) &:= \frac{1}{t-s} \int_s^t (V_{s,\lambda} - 1) \langle u, G(\lambda) v \rangle d\lambda \mathbf{e}(0).\end{aligned}$$

Note that any  $\phi \in \Gamma$  can be written as  $\phi = \phi^{(0)} \oplus \phi^{(1)} \oplus \dots$ , where  $\phi^{(n)}$  is in the  $n$ -fold symmetric tensor product  $L^2(\mathbb{R}_+, \mathbf{k})^{\otimes n} \equiv L^2(\Sigma_n) \otimes \mathbf{k}^{\otimes n}$ . Here  $\Sigma_n$  is the  $n$ -simplex  $\{\underline{t} = (t_1, t_2, \dots, t_n) : 0 \leq t_1 < t_2 < \dots < t_n < \infty\}$ .

*Lemma 5.1.* Let  $u, v \in \mathbf{h}$  and let  $C_\tau = 4e^\tau \sup\{\|L(\lambda)\|^2 + \|G(\lambda)\|^2 : 0 \leq \lambda \leq \tau\}$ . Then for any  $0 \leq s \leq t \leq \tau$ ,

(i)

$$(5.10) \quad \|(V_{s,t} - 1)v\mathbf{e}(0)\|^2 \leq C_\tau |t - s| \|v\|^2.$$

(ii) For any  $u \in \mathbf{h}$

$$\begin{aligned}& \left\| \sum_{j \geq 1} \int_s^t V_{s,\lambda} \langle u, L_j(\lambda) v \rangle a_j^\dagger(d\lambda) \mathbf{e}(0) \right\|^2 \\ & \leq \|u\|^2 \left\| \sum_{j \geq 1} \int_s^t V_{s,\lambda} L_j(\lambda) d\lambda v \otimes \mathbf{e}(0) \right\|^2.\end{aligned}$$

*Proof.* (i) By estimates of quantum stochastic integration (Proposition 27.1, [14])

$$\begin{aligned}& \|(V_{s,t} - 1)v\mathbf{e}(0)\|^2 \\ &= \left\| \sum_{j \geq 1} \int_s^t V_{s,\lambda} L_j(\lambda) a_j^\dagger(d\lambda) v\mathbf{e}(0) + \int_s^t V_{s,\lambda} G(\lambda) d\lambda v\mathbf{e}(0) \right\|^2 \\ &\leq 2e^\tau \int_s^t \left\{ \sum_{j \geq 1} \|L_j(\lambda) v\|^2 + \|G(\lambda) v\|^2 \right\} d\lambda \\ &\leq 2e^\tau \|v\|^2 \int_s^t \{\|L(\lambda)\|^2 + \|G(\lambda)\|^2\} d\lambda \\ &= \|v\|^2 C_\tau |t - s|.\end{aligned}$$

(ii) For any  $\phi$  in the Fock space  $\Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ ,

$$\begin{aligned} & |\langle \phi, \sum_{j \geq 1} \int_s^t V_{s,\lambda}(u, L_j(\lambda)v) a_j^\dagger(d\lambda) \mathbf{e}(0) \rangle|^2 \\ &= |\langle u \otimes \phi, \{ \sum_{j \geq 1} \int_s^t V_{s,\lambda} L_j(\lambda) a_j^\dagger(d\lambda) \} v \mathbf{e}(0) \rangle|^2 \\ &\leq \|u \otimes \phi\|^2 \| \{ \sum_{j \geq 1} \int_s^t V_{s,\lambda} L_j(\lambda) a_j^\dagger(d\lambda) \} v \mathbf{e}(0) \|^2. \end{aligned}$$

Since  $\phi$  is arbitrary, the first inequality follows. Thus further by the estimates of quantum stochastic integration

$$\begin{aligned} & \left\| \sum_{j \geq 1} \int_s^t V_{s,\lambda}(u, L_j(\lambda)v) a_j^\dagger(d\lambda) \mathbf{e}(0) \right\|^2 \leq 2e^\tau \|u\|^2 \int_s^t \sum_{j \geq 1} \|V_{s,\lambda} L_j(\lambda)v\|^2 d\lambda \\ (5.11) \quad & \leq 2e^\tau \|u\|^2 \int_s^t \|L(\lambda)v\|^2 d\lambda \leq |t-s| \|u\|^2 \|v\|^2 C_\tau. \end{aligned}$$

□

*Lemma 5.2.* Let  $C'_\tau = 4e^{2\tau} \sup\{\|(L(\alpha) \otimes 1)L(\lambda)\|^2 + \|(G(\alpha) \otimes 1)L(\lambda)\|^2 : \alpha, \lambda \in [0, \tau]\}$  and  $C''_\tau = 4e^{2\tau} \sup\{\|(L(\alpha) \otimes 1)G(\lambda)\|^2 + \|(G(\alpha) \otimes 1)G(\lambda)\|^2 : \alpha, \lambda \in [0, \tau]\}$ . Then for any  $u, v \in \mathbf{h}$ ,  $0 \leq s \leq t \leq \tau$

$$(i) \quad \|(V_{s,t} - 1)(u, v) \mathbf{e}(0)\|^2 \leq C_\tau \|u\|^2 \|v\|^2 |t-s|.$$

$$(ii) \quad \sup\{\|\zeta(s, t, u, v)\|^2 : 0 \leq s \leq t \leq \tau\} \leq C'_\tau \|u\|^2 \|v\|^2 \text{ and } \|\zeta(s, t, u, v)\| \leq \sqrt{C''_\tau |t-s|} \|u\| \|v\|.$$

(iii) For any  $\phi \in \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ ,  $\lim_{t \downarrow s} \langle \phi, \zeta(s, t, u, v) \rangle = 0$  and

$$\lim_{t \downarrow s} \langle \phi, \gamma(s, t, u, v) \rangle = \sum_{j \geq 1} \langle u, L_j(s)v \rangle \overline{\phi_j^{(1)}(s)} = \langle \phi^{(1)}(s), \eta_s(u, v) \rangle, \text{ a.e. } s \geq 0.$$

*Proof.* (i) By (5.4) and (5.11) we have

$$\begin{aligned} & \|(V_{s,t} - 1)(u, v) \mathbf{e}(0)\|^2 \\ &= \left\| \sum_{j \geq 1} \int_s^t V_{s,\lambda}(u, L_j(t)v) a_j^\dagger(d\lambda) \mathbf{e}(0) + \int_s^t V_{s,\lambda}(u, G(\lambda)v) \mathbf{e}(0) d\lambda \right\|^2 \\ &\leq 2 \left\| \sum_{j \geq 1} \int_s^t V_{s,\lambda}(u, L_j(\lambda)v) a_j^\dagger(d\lambda) \mathbf{e}(0) \right\|^2 + \left[ \int_s^t \|V_{s,\lambda}(u, G(\lambda)v) \mathbf{e}(0)\| d\lambda \right]^2 \\ &\leq 4e^\tau \|u\|^2 \|v\|^2 \int_s^t [\|L(\lambda)\|^2 + \|G(\lambda)\|^2] d\lambda \\ &\leq C_\tau \|u\|^2 \|v\|^2 |t-s|. \end{aligned}$$



(ii) By inequalities (5.11) we have

$$\begin{aligned} \|\zeta(s, t, u, v)\|^2 &= \frac{1}{|t-s|^2} \left\| \sum_{j \geq 1} \int_s^t (V_{s,\lambda} - 1)(u, L_j(\lambda)v) a_j^\dagger(d\lambda) \mathbf{e}(0) \right\|^2 \\ &\leq \frac{2e^\tau \|u\|^2}{|t-s|^2} \int_s^t \sum_{j \geq 1} \|(V_{s,\lambda} - 1)L_j(\lambda)v \mathbf{e}(0)\|^2 d\lambda. \end{aligned}$$

Now as in Lemma 5.1 (i), the above quantity can be estimated by

$$\begin{aligned} &\leq \frac{4e^{2\tau} \|u\|^2}{|t-s|^2} \int_s^t \sum_{j \geq 1} \left\{ \int_s^\lambda \sum_{i \geq 1} \|L_i(\alpha)L_j(\lambda)v\|^2 + \|G(\alpha) L_j(\lambda) v\|^2 \right\} d\alpha d\lambda \\ &\leq \frac{4e^{2\tau} \|u\|^2}{|t-s|^2} \int_s^t \int_s^\lambda \left\{ \|(L(\alpha) \otimes 1)L(\lambda)v\|^2 + \|(G(\alpha) \otimes 1) L(\lambda) v\|^2 \right\} d\alpha d\lambda, \end{aligned}$$

which leads to the statement.

Also we have

$$\begin{aligned} \|\zeta(s, t, u, v)\| &= \frac{1}{|t-s|} \left\| \int_s^t (V_{s,\lambda} - 1)(u, G(\lambda)v) d\lambda \mathbf{e}(0) \right\| \\ &\leq \frac{1}{|t-s|} \int_s^t \|(V_{s,\lambda} - 1)(u, G(\lambda)v) \mathbf{e}(0)\| d\lambda. \end{aligned}$$

Thus, similarly as above, the estimate follows.

(iii) For any  $f \in L^2(\mathbb{R}_+, \mathbf{k})$ . Let us consider

$$\begin{aligned} \langle \mathbf{e}(f), \zeta(s, t, u, v) \rangle &= \langle \mathbf{e}(f), \frac{1}{t-s} \sum_{j \geq 1} \int_s^t (V_{s,\lambda} - 1)(u, L_j(\lambda)v) a_j^\dagger(d\lambda) \mathbf{e}(0) \rangle \\ &= \frac{1}{t-s} \sum_{j \geq 1} \int_s^t \overline{f_j(\lambda)} \langle \mathbf{e}(f), (V_{s,\lambda} - 1)(u, L_j(\lambda)v) \mathbf{e}(0) \rangle d\lambda \\ &= \frac{1}{t-s} \int_s^t R(s, \lambda) d\lambda, \end{aligned}$$

where  $G(s, \lambda) = \sum_{j \geq 1} \overline{f_j(\lambda)} \langle \mathbf{e}(f), (V_{s,\lambda} - 1)(u, L_j(\lambda)v) \mathbf{e}(0) \rangle$ . Note that the complex valued function  $R(s, \lambda)$  is locally integrable in  $\lambda$  and continuous in  $s$  and therefore it makes sense to talk about  $R(s, s)$  which is 0. So we get

$$\lim_{t \downarrow s} \langle \mathbf{e}(f), \zeta(s, t, u, v) \rangle = 0.$$

Since  $\zeta(s, t, u, v)$  is uniformly bounded in  $s, t$

$$\lim_{t \downarrow s} \langle \phi, \zeta(s, t, u, v) \rangle = 0, \forall \phi \in \Gamma.$$

We also have

$$(5.12) \quad \langle \phi, \gamma(s, t, u, v) \rangle = \frac{1}{t-s} \sum_{j \geq 1} \int_s^t \langle u, L_j(\lambda)v \rangle \overline{\phi_j^{(1)}(\lambda)} d\lambda.$$

Since

$$\left| \sum_{j \geq 1} \langle u, L_j(\lambda)v \rangle \overline{\phi_j^{(1)}(\lambda)} \right|^2 \leq \|u\|^2 \sum_{j \geq 1} \|L_j(\lambda)v\|^2 |\phi_j^{(1)}(\lambda)|^2 \leq C_\tau \|v\|^2 \|\phi^{(1)}(\lambda)\|^2,$$

the function  $\sum_{j \geq 1} \langle u, L_j(\lambda)v \rangle \overline{\phi_j^{(1)}}(\lambda)$  is in  $L^2$  and hence locally integrable. Thus we get

$$\lim_{t \downarrow s} \langle \phi, \gamma(s, t, u, v) \rangle = \sum_{j \geq 1} \langle u, L_j(s)v \rangle \overline{\phi_j^{(1)}}(s) = \langle \phi^{(1)}(s), \eta_s(u, v) \rangle \text{ a.e. } s \geq 0.$$

□

**Lemma 5.3.** For  $n \geq 1$ ,  $\underline{s} \in \Sigma_n$  and  $u_k, v_k \in \mathbf{h} : k = 1, 2, \dots, n, \phi \in \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$  and disjoint  $[s_k, t_k]$ ,

- (i)  $\lim_{\underline{t} \downarrow \underline{s}} \langle \phi, \prod_{k=1}^n M(s_k, t_k, u_k, v_k) \mathbf{e}(0) \rangle = 0$ ,  
 where  $M(s_k, t_k, u_k, v_k) = \frac{(V_{s_k, t_k} - 1)}{t_k - s_k} (u_k, v_k) - \rho(s_k, t_k, u_k, v_k) - \gamma(s_k, t_k, u_k, v_k)$  and  $\lim_{\underline{t} \downarrow \underline{s}}$  means  $t_k \downarrow s_k$  for each  $k$ .
- (ii)  $\lim_{\underline{t} \downarrow \underline{s}} \langle \phi, \otimes_{k=1}^n \gamma(s_k, t_k, u_k, v_k) \rangle = \langle \phi^{(n)}(s_1, s_2, \dots, s_n), \eta_{s_1}(u_1, v_1) \otimes \dots \otimes \eta_{s_n}(u_n, v_n) \rangle$ .

*Proof.* (i) First note that  $M(s, t, u, v)\mathbf{e}(0) = \zeta(s, t, u, v) + \varsigma(s, t, u, v)$ . So by the above observations in Lemma 5.2,  $\{M(s, t, u, v)\mathbf{e}(0)\}$  is uniformly bounded in  $s, t \leq \tau$  and  $\lim_{t \downarrow s} \langle \mathbf{e}(f), M(s, t, u, v)\mathbf{e}(0) \rangle = 0, \forall f \in L^2(\mathbb{R}_+, \mathbf{k})$ . Since the intervals  $[s_k, t_k]$ 's are disjoint for different  $k$ 's,

$$\langle \mathbf{e}(f), \prod_{k=1}^n M(s_k, t_k, u_k, v_k) \mathbf{e}(0) \rangle = \prod_{k=1}^n \langle \mathbf{e}(f_{[s_k, t_k]}), M(s_k, t_k, u_k, v_k) \mathbf{e}(0) \rangle$$

and thus  $\lim_{\underline{t} \downarrow \underline{s}} \langle \mathbf{e}(f), \prod_{k=1}^n M(s_k, t_k, u_k, v_k) \mathbf{e}(0) \rangle = 0$ .

Since  $\prod_{k=1}^n M(s_k, t_k, u_k, v_k) \mathbf{e}(0)$  is uniformly bounded in  $s_k, t_k$  requirement follows for  $\phi \in \Gamma$ .

(ii) It can be proved similarly as part (iii) of the previous Lemma.

□

**Lemma 5.4.** Let  $\phi \in \Gamma$  be such that

$$(5.13) \quad \langle \phi, \psi \rangle = 0, \quad \forall \psi \in \mathcal{S}'.$$

Then we have

- (i)  $\phi^{(0)} = 0$  and  $\phi^{(1)} = 0$ ,
- (ii) for any  $n \geq 0$ ,  $\phi^{(n)} = 0$ ,
- (iii) the set  $\mathcal{S}'$  is total in the Fock space  $\Gamma$ .

*Proof.* (i) For any  $s \geq 0$ ,  $V_{s,s} = 1_{\mathbf{h} \otimes \Gamma}$  and so, in particular, (5.13) gives, for any  $u, v \in \mathbf{h}$ ,

$$0 = \langle \phi, V_{s,s}(u, v)\mathbf{e}(0) \rangle = \langle u, v \rangle \overline{\phi^{(0)}}$$

and hence  $\phi^{(0)} = 0$ .

(ii) By (5.13),  $\langle \phi, [V_{s,t} - 1](u, v)\mathbf{e}(0) \rangle = 0$  for any  $0 \leq s < t \leq \tau < \infty$  and  $u, v \in \mathbf{h}$ . By HP equation (5.4) and part (iii) of Lemma 5.2, we have

$$\begin{aligned} 0 &= \lim_{t \downarrow s} \frac{1}{t - s} \langle \phi, [V_{s,t} - 1](u, v)\mathbf{e}(0) \rangle \\ &= \sum_{j \geq 1} \langle u, L_j(s)v \rangle \overline{\phi_j^{(1)}}(s) \\ &= \langle \phi^{(1)}(s), \eta_s(u, v) \rangle. \end{aligned}$$

So  $\langle \phi^{(1)}(s), \eta_s(u, v) \rangle = 0$  for any  $u, v \in \mathbf{h}$  for almost every  $s$ . Since  $\{\eta_s(u, v) : u, v \in \mathbf{h}\}$  is total in  $\mathbf{k}_s$ , it follows that  $\phi^{(1)}(s) = 0 \in \mathbf{k}_s$  for almost every  $0 \leq s \leq \tau$ , i.e.,  $\phi^{(1)} = 0$ .

(iii) We prove this by induction. The result is already proved for  $n = 0, 1$ . For  $n \geq 2$ , assume as induction hypothesis that for all  $m \leq n - 1$ ,  $\phi^{(m)}(\underline{s}) = 0$ , for almost every  $\underline{s} \in \Sigma_m$  ( $s_i \leq \tau$  for  $i = 1, 2, \dots, m$ ). To show that  $\phi^{(n)} = 0$ , we note that by a similar argument as in [16],

$$\langle \phi^{(n)}(s_1, s_2, \dots, s_n), \eta_{s_1}(u_1, v_1) \otimes \dots \otimes \eta_{s_n}(u_n, v_n) \rangle = 0.$$

for almost every  $\underline{s} \in \Sigma_n$  ( $s_i \leq \tau$ ). Since  $\{\eta_s(u, v) : u, v \in \mathbf{h}\}$  is total in  $\mathbf{k}_s$ , it follows that  $\phi^{(n)}(s_1, s_2, \dots, s_n) = 0 \in \mathbf{k}_{s_1} \otimes \dots \otimes \mathbf{k}_{s_n}$  for almost every  $(s_1, s_2, \dots, s_n) \in \Sigma_n$ .  $\square$

**5.3. Unitary Equivalence.** We shall now prove (ii) in Theorem 5.1 that the unitary evolution  $\{U_{s,t}\}$  on  $\mathbf{h} \otimes \mathcal{H}$  is unitarily equivalent to the unitary solution  $\{V_{s,t}\}$  of HP equation (5.4). To prove this we need the following two results.

**Lemma 5.5.** *Let  $U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\Omega$  and  $U_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w})\Omega$  be in  $\mathcal{S}$ , where  $\underline{u}, \underline{z} \in \mathbf{h}^{\otimes n}$ . Then there exist an integer  $m \geq 1$ ,  $\underline{a} = (a_1, a_2, \dots, a_m)$ ,  $\underline{b} = (b_1, b_2, \dots, b_m)$  with  $0 \leq a_1 \leq b_1 \leq \dots \leq a_m \leq b_m < \infty$ , partition  $R_1 \cup R_2 \cup R_3 = \{1, \dots, m\}$  with  $|R_i| = m_i$ , family of vectors  $x_{k_l}, g_{k_i} \in \mathbf{h}$  and  $y_{k_l}, h_{k_i} \in \mathbf{h}$  for  $l \in R_1 \cup R_2$  and  $i \in R_2 \cup R_3$  such that*

$$U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v}) = \sum_{\underline{k}} \prod_{l \in R_1 \cup R_2} U_{a_l, b_l}(x_{k_l}, y_{k_l}),$$

$$U_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w}) = \sum_{\underline{k}} \prod_{l \in R_2 \cup R_3} U_{a_l, b_l}(g_{k_l}, h_{k_l}).$$

*Proof.* It follows from the evolution hypothesis of  $\{U_{s,t}\}$  that for  $r \in [s, t]$  and a complete orthonormal basis  $\{f_j\} \in \mathbf{h}$  we can write  $U_{s,t}(u, v) = \sum_{j \geq 1} U_{s,r}(u, f_j) U_{r,t}(f_j, v)$ .  $\square$

*Remark 5.6.* Since the family of unitary operators  $\{V_{s,t}\}$  on  $\mathbf{h} \otimes \Gamma$  enjoy all the properties satisfy by family of unitary operators  $\{U_{s,t}\}$  on  $\mathbf{h} \otimes \mathcal{H}$ , the above lemma also hold if we replace  $U_{s,t}$  by  $V_{s,t}$ .

**Lemma 5.7.** *For  $U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\Omega, U_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w})\Omega \in \mathcal{S}$ , we have*

$$(5.14) \quad \langle U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\Omega, U_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w})\Omega \rangle = \langle V_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0), V_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w})\mathbf{e}(0) \rangle.$$

*Proof.* The proof of (5.14) is very similar to that in [16]. In fact, for

$$0 \leq s \leq t < \infty, \langle U_{s,t}(u, v)\Omega, U_{s,t}(p, w)\Omega \rangle = \langle p, Z_{s,t}(|w\rangle\langle v|)u \rangle$$

while

$$\langle V_{s,t}(u, v)\mathbf{e}(0), V_{s,t}(p, w)\mathbf{e}(0) \rangle = \langle p, \tilde{Z}_{s,t}(|w\rangle\langle v|)u \rangle$$

but  $\tilde{Z}_{s,t} = Z_{s,t}$ .  $\square$

Now defining a map  $\Xi : \mathcal{H} \rightarrow \Gamma$  by sending  $U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\Omega \in \mathcal{S}$  to  $V_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0) \in \mathcal{S}'$ , as in [16], we can establish unitary equivalence of HP evolution  $V_{s,t}$  with the evolution  $U_{s,t}$  we started with.

## 6. Appendix

Let  $X$  be a complex separable Banach space. Consider the Banach space

$$\widehat{X} = L^1(\mathbb{R}_+, X) = \left\{ f : \mathbb{R}_+ \rightarrow X \text{ a Lebesgue measurable, } \|f\| := \int_{\mathbb{R}_+} \|f(\tau)\| d\tau < \infty \right\}$$

and define shift operators  $U_t$  on  $\widehat{X}$  given by

$$U_t f(\tau) := \begin{cases} 0 & \text{if } \tau < t, \\ f(\tau - t) & \text{if } \tau \geq t. \end{cases}$$

Then for each  $t \geq 0$ ,  $U_t$  is an isometry and  $\{U_t\}$  is a strongly continuous semigroup with generator  $P = -\frac{d}{dt}$  with domain

$$\mathcal{D}(P) = \left\{ f \in \widehat{X} : f(0) = 0, f \text{ is absolutely continuous, } f' \in \widehat{X} \right\}.$$

The adjoint of this semigroup is given by  $U_t^* f(\tau) = f(\tau + t)$ .

Let  $\{S_{s,t} : 0 \leq s \leq t < \infty\}$  be an evolution of contraction operators in  $\mathcal{B}(X)$ . With further conditions on  $S_{s,t}$  we have the following result

**Theorem 6.1.** *Let  $\{S_{s,t} : 0 \leq s \leq t < \infty\}$  be a contractive evolution in  $\mathcal{B}(X)$  such that  $\|S_{s,t} - 1\| \leq C|t - s|$ , where  $C$  is independent of  $s, t$ . Then there exists a Lebesgue measurable function  $G : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$  such that  $G$  is locally essentially norm bounded and*

$$S_{s,t} - 1 = \int_s^t S_{s,\tau} G(\tau) d\tau.$$

*Proof.* Consider the family of operators  $\mathcal{S}_t$  in  $\widehat{X}$  define by

$$\mathcal{S}_t f(\tau) = S_{\tau, \tau+t} f(\tau + t) = S_{\tau, \tau+t} U_t^* f(\tau).$$

Then  $\mathcal{S}_t$  is a contractive strongly continuous semigroup on  $\widehat{X}$ . To prove the contractivity, for  $t \geq 0$ , consider

$$\|\mathcal{S}_t f\| \leq \int_{\mathbb{R}_+} \|S_{\tau, \tau+t} f(\tau + t)\| d\tau = \int_{\mathbb{R}_+} \|f(\tau + t)\| d\tau \leq \|f\|.$$

Also, we have the semigroup property:

$$\begin{aligned} \mathcal{S}_t \mathcal{S}_s f(\tau) &= S_{\tau, \tau+t} (\mathcal{S}_s f)(\tau + t) = S_{\tau, \tau+t} S_{\tau+t, \tau+t+s} f(\tau + t + s) \\ &= S_{\tau, \tau+t+s} f(\tau + t + s) = \mathcal{S}_{t+s} f(\tau). \end{aligned}$$

To prove strong continuity of  $\mathcal{S}_t$ , we first consider, for  $g \in \mathcal{C}_0^\infty(\mathbb{R}_+, X)$

$$\begin{aligned} \|(\mathcal{S}_t - 1)g\| &= \int_{\mathbb{R}_+} \|(\mathcal{S}_t - 1)g(\tau)\| d\tau = \int_{\mathbb{R}_+} \|(S_{\tau, \tau+t} U_t^* - 1)g(\tau)\| d\tau \\ &\leq \int_{\mathbb{R}_+} \|(S_{\tau, \tau+t} - 1)U_t^* g(\tau)\| d\tau + \int_{\mathbb{R}_+} \|g(t + \tau) - g(\tau)\| d\tau \\ &= C\|g\|t + \int_{\mathbb{R}_+} \|g(t + \tau) - g(\tau)\| d\tau. \end{aligned}$$

We can use dominated convergence theorem to take limit of the second term as  $g$  is compactly supported continuous function and concluded that  $\|(\mathcal{S}_t - 1)g\|$  converges to 0. The  $\mathcal{S}_t$  is strongly continuous follows from density of  $\mathcal{C}_0^\infty(\mathbb{R}_+, X)$ . So there exists a

densely defined, closed maximally dissipative operator  $\mathcal{G}$  which is the generator of the semigroup  $\mathcal{S}_t$ . Thus we have

$$\mathcal{G}f = \lim_{t \rightarrow 0} \frac{\mathcal{S}_t f - f}{t}$$

in  $\widehat{X}$ -norm for each  $f \in \mathcal{D}(\mathcal{G})$  and hence there exists a sequence  $t_n$  tending to 0 such that for almost every  $\tau$

$$\begin{aligned} \mathcal{G}f(\tau) &= \lim_{t_n \rightarrow 0} \frac{\mathcal{S}_{t_n} f(\tau) - f(\tau)}{t_n} \\ &= \lim_{t_n \rightarrow 0} \frac{S_{\tau, \tau+t_n} f(\tau+t_n) - f(\tau)}{t_n} \\ &= \lim_{t_n \rightarrow 0} \left\{ \frac{(S_{\tau, \tau+t_n} - 1)(f(\tau+t_n) - f(\tau))}{t_n} + \frac{(S_{\tau, \tau+t_n} - 1)f(\tau)}{t_n} + \frac{f(\tau+t_n) - f(\tau)}{t_n} \right\}. \end{aligned}$$

Since  $\left\| \frac{(S_{\tau, \tau+t_n} - 1)}{t_n} \right\| \leq C$ , for  $f \in \mathcal{D}(\mathcal{G}) \cap \mathcal{D}(P)$ ,  $\lim_{t_n \rightarrow 0} \frac{\mathcal{S}_{t_n} f(\tau) - f(\tau)}{t_n} = \mathcal{G}f(\tau) - Pf(\tau)$  for almost every  $\tau$ . Define it to be  $G(\tau)f(\tau)$ . On the other hand, for any  $t, \beta, \sigma \geq 0$

$$\frac{1}{\sigma} \int_{t+\beta}^{t+\beta+\sigma} S_{t,\tau} d\tau - \frac{1}{\sigma} \int_t^{t+\sigma} S_{t,\tau} d\tau = \int_t^{t+\beta} S_{t,\tau} \frac{S_{\tau, \tau+\sigma} - 1}{\sigma} d\tau.$$

Therefore, we have  $\lim_{\sigma \rightarrow 0} \int_t^{t+\beta} S_{t,\tau} \frac{S_{\tau, \tau+\sigma} - 1}{\sigma} d\tau = S_{t, t+\beta} - 1$  by continuity. Note that the domain  $\mathcal{D}(\mathcal{G})$  contains  $\mathcal{C}_0^\infty(\mathbb{R}_+, X)$ . Let  $g \in \mathcal{C}_0^\infty(\mathbb{R}_+, X)$  such that for any  $t \in (t_1, t_2)$ ,  $g(t) = x$  for some  $x \in X$ , such a  $g \in \mathcal{D}(\mathcal{G}) \cap \mathcal{D}(P)$ . Therefore, for almost every  $t \in (t_1, t_2)$ ,

$$(6.1) \quad G(t)x = \lim_{\beta \rightarrow 0} \frac{S_{t, t+\beta} g(t) - g(t)}{\beta} = \lim_{\beta \rightarrow 0} \frac{1}{\beta} \int_t^{t+\beta} S_{t,\tau} (\mathcal{G}g(\tau)) d\tau = \mathcal{G}g(t).$$

Since we have  $\frac{S_{t, t+\beta} x - x}{\beta}$  is continuous in  $t$ , in particular measurable and as a point wise limit of measurable functions  $t \mapsto G(t)x$  is Lebesgue measurable.

To see the boundedness of  $G(t)$ , consider the following. By the identity (6.1),  $G(t)$  is defined on whole of the Banach space  $X$ . So it is enough to show that  $G(t)$  is closable and use the close graph theorem. Let  $u_n$  converge to 0 such that  $G(t)u_n$  converges to  $v$ . Let us consider a sequence of vectors  $g_n \in \mathcal{C}_0^\infty(\mathbb{R}_+, X)$  taking value  $u_n$  in an interval  $(t_1, t_2)$  containing  $t$  and converging to 0 in  $\widehat{X}$ . We can choose  $g_n \in \widehat{X}$  such that  $Pg_n$  and  $\mathcal{G}g_n$  converges and  $\mathcal{G}g_n(t) = G(t)u_n$  for each  $t \in (t_1, t_2)$  and hence closability of  $\mathcal{G}$  gives that  $\mathcal{G}g_{n_k}$ , for a sub sequence, converges to 0 point wise. Since  $\mathcal{G}g_{n_k}(t) = G(t)u_{n_k}$ , the limit  $v = 0$ . Therefore,  $G(t)$  is closable and defined everywhere proving that it is bounded for almost all  $t$ . Note that

$$(6.2) \quad G(t)x = \lim_{n \rightarrow \infty} \frac{S_{t, t+t_n} - 1}{t_n} x$$

and by Assumption  $\left\| \frac{S_{t, t+t_n} - 1}{t_n} x \right\| \leq C\|x\|$ . Thus we have

$$\|G(t)\| \leq \liminf_{n \rightarrow \infty} \left\| \frac{S_{t, t+t_n} - 1}{t_n} \right\| \leq C.$$

□

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